Pursuit-Evasion Between Two Realistic Aircraft

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Pursuit-evasion problems between two aircraft in coplanar motion are analyzed by the differential dynamic programming method. For each aircraft a realistic variable-speed model is used where the longitudinal acceleration depends on speed, turn rate, and throttle setting. The turn rate is constrained by structural as well as stall limits. The pursuer has the turn rate and throttle setting as input controls, whereas the evader only uses the turn rate. In addition, the pursuer's optimal initial speed can be obtained from the algorithm used. The performance objective in the pursuit-evasion problem is the separation distance at a specified final time. Solutions of this nontrivial problem are calculated for a range of parameter settings. The results show that optimal setting of the throttle and optimal choice of the initial velocity can render considerable improvement for the pursuer, relative to nonoptimal strategies.

Introduction

DIFFERENTIAL game problems consisting of two aircraft with variable speeds in coplanar motion are analyzed. The formulation is made as a pursuit-evasion problem, with an arbitrary initial condition and the roles fixed a priori. In aiming to solve realistic aerial combat problems, an important fact is that the longitudinal acceleration of each aircraft depends on speed, turn rate, and throttle setting. In addition, the turn rate is constrained by structural as well as stall limits. Results of this class of problem occur sparsely in the literature, where often low-order, constant-speed dynamics models and approximate methods[1] are used. With the use of an effective optimization algorithm, several examples are solved in the present paper.

The game usually will end up in a coplanar tail chase pursuit-evasion situation similar to the one studied in Ref. 2. Other studies of the general combat situation dealt with herein are found in Refs. 1 and 3 to some extent. In Ref. 1, suboptimal control laws are derived which can be applied in a nontail chase combat problem. The optimization algorithm used in our study is a modified differential dynamic programming (DDP) method for solving general optimal open-loop differential games.[4]

Optimization of the differential game stated will make full use of the aircraft performance. From the optimal solution of the differential game concise conclusions can be drawn, compared to conclusions drawn from, for example, simulation results. The differential game is assumed to take place in a horizontal plane. This is an approximation to real aerial combat where a high-performance aircraft, in particular, might make vertical maneuvers. However, the limitation to the horizontal plane emphasizes the use of the throttle setting as well as the interplay between the turning capability and the acceleration or high-speed performance.

The measure of the game is the separation distance at a given final time. The relative importance of turning and speed capabilities can be seen using this measure. The final time value is a game parameter which influences the results; for example, games of short duration use hard controls. When analyzing the game for several initial conditions, some such conditions will be more demanding on the turn or speed capability of the players. First we have to find a relevant and practically motivated situation where the aircraft performance and controls might be examined. Such a situation is the head-on encounter used herein. For these purposes an effective optimization method is needed. Simulations and suboptimal control laws such as in Ref. 1 have to be evaluated before being used in this way.

The algorithm described in Refs. 4 and 5 is a DDP method complemented with a convergence control parameter (CCP) technique, which makes it possible to influence each of the control variables individually. The method used in Ref. 4 and in this paper is a first-order DDP method. The DDP methods originate from Ref. 6, particularly the second-order method.

The purpose of this paper is to investigate the use of certain features of an aircraft in a combat situation. Also, the efficiency of the optimization method used will be demonstrated.

Equations of Motion

The motion of two aircraft at equal constant altitudes in a fixed frame is modeled by the kinematic equations given the location of the pursuing (x_p, y_p) and evading (x_e, y_e) aircraft (see Fig. 1)

$$\dot{x}_p = v_p \cos\phi_p \tag{1}$$

$$\dot{y}_p = v_p \sin\phi_p \tag{2}$$

$$\dot{x}_e = v_e \cos\phi_e \tag{3}$$

$$\dot{y}_e = v_e \sin\phi_e \tag{4}$$

where the headings (ϕ_p, ϕ_e) and speeds (v_p, v_e) evolve according to

$$\dot{\phi}_p = u_p \tag{5}$$

$$\dot{\phi}_e = u_e \tag{6}$$

$$\dot{v}_p = \left[T_p(v_p, h)u_T - D_p(v_p, h, u_p)\right]/m_p \tag{7}$$

$$\dot{v}_e = \left[T_e(v_e, h) - D_e(v_e, h, u_e)\right]/m_e \tag{8}$$

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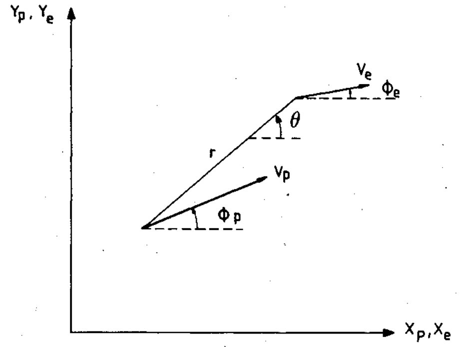

Fig. 1 Model geometry.

where u_p and u_e are the pursuer and evader turn rates; T_p and T_e the thrusts; D_p and D_e the aerodynamic drags; m_p and m_e the masses (assumed constant); h the altitude (assumed fixed); and u_T the throttle setting for the pursuer, $0 \leq u_T \leq 1$. There is no need for an evader throttling capability in the case studied. Furthermore, the effects of the angle of attack are neglected with respect to the thrust deflection. In vector notation the state vector will be $x = (x_p, y_p, x_e, y_e, \phi_p, \phi_e, v_p, v_e)^T$ and the control vector will be $u = (u_p, u_T, u_e)^T$.

The drag is split into a zero-lift component and an induced-lift component, where the latter depends on the turn rate

$$D = qS\left(C_{D_0} + K_D C_L^2\right) \tag{9}$$

where C_{D_0} and K_D are aerodynamic coefficients depending in a nonlinear manner on Mach number, S is the wing area, q the dynamic pressure, and C_L the lift coefficient,

$$q = \tfrac{1}{2}\rho(h)v^2 \tag{10}$$

$$C_L = \frac{mg}{Sq}\sqrt{1 + (u_i v/g)^2}, \quad i = 1, 3 \tag{11}$$

The air density $\rho(h)$ depends on the selected altitude h in accordance with the ICAO standard atmosphere.

In the computer program used, the nonlinear thrusts and aerodynamic coefficients are calculated by fitting smooth functions to the realistic data given by plots. This fitting procedure has been performed in order to obtain smooth partial derivatives for the optimization algorithm.

The performance of an aircraft is often represented by the specific excess power (SEP), defined as

$$\text{SEP} = \frac{v\dot{v}}{g} = \frac{T - D}{mg}v \tag{12}$$

The pursuer used in the examples below has $m = 9000$ kg and $S = 30$ m^2. The SEP of the pursuer is plotted in Fig. 2, representing very good performance.

The turn rates are constrained by the maximum lift coefficient at low speeds and by the structural limit at high speeds. The maximum turn rate, $\hat{u}_i$, vs speed is depicted in Fig. 3.

The stall limit gives

$$|u_i| \leq \hat{u}_i = g\sqrt{K^2 v^2 - 1/v^2}, \quad i = 1, 3, \quad v_{\min} \leq v \leq v_c \tag{13}$$

$$K = 0.5\rho S C_{L_{\max}}/(mg) \tag{14}$$

The structural limit gives

$$|u_i| \leq \hat{u}_i = g\sqrt{n^2 - 1}/v, \quad i = 1, 3, \quad v > v_c \tag{15}$$

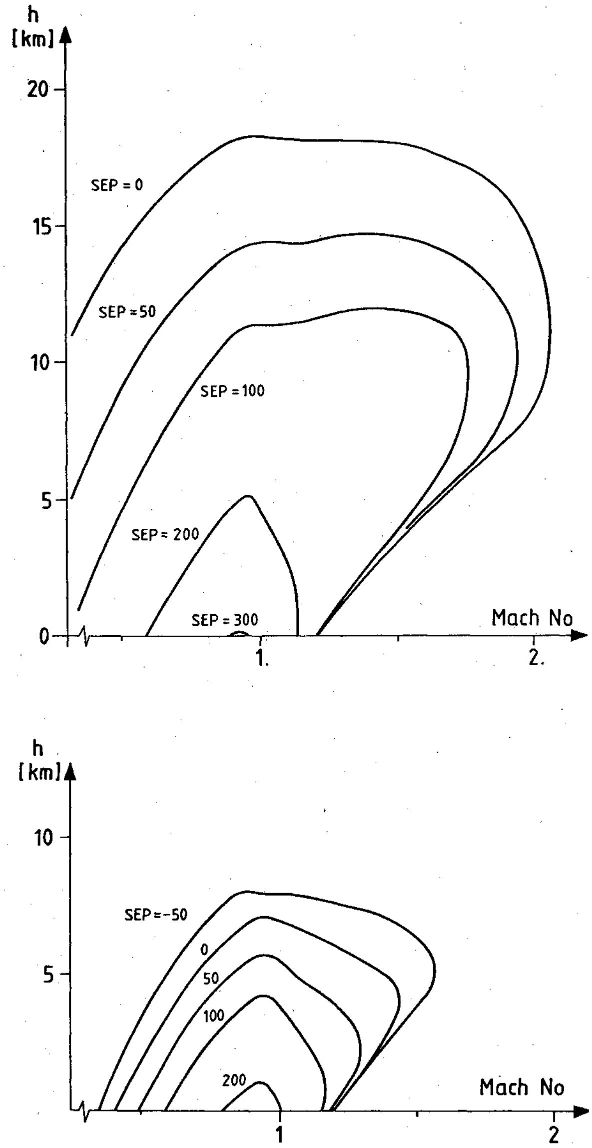

Fig. 2 SEP (m/s) vs Mach number and altitude.
a) Load factor = 1.
b) Load factor = 5.

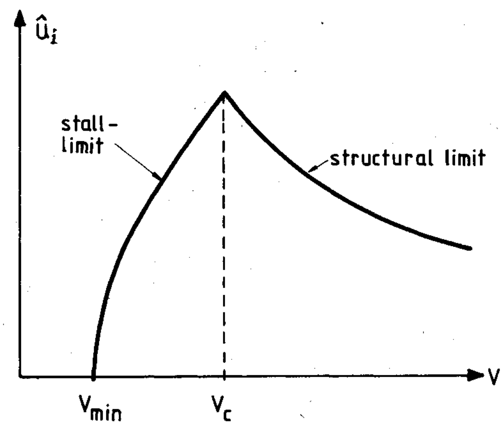

Fig. 3 Maximum turn rate vs speed at constant altitude.

where n is the maximal load factor, g the acceleration due to gravity, $C_{L_{max}}$ the maximum lift coefficient, and v_c denotes the corner speed

$$v_c = \sqrt{n/K} \tag{16}$$

Optimization Problem

Performance Index

Let the final time t_f be fixed and the payoff of the game be the distance r_f between the two aircraft at t_f, to be minimized by the pursuer and maximized by the evader. Hence, we define the performance index as

$$V = r_f \tag{17}$$

The optimization of this quantity shall be achieved subject to the state equations (1-8) and the constraints on the throttle setting and turn rates.

Optimality Conditions

The Hamiltonian corresponding to the dynamic equations is

$$H = v_p\left(V_{x_p}\cos\phi_p + V_{y_p}\sin\phi_p\right) + v_e\left(V_{x_e}\cos\phi_e + V_{y_e}\sin\phi_e\right) + V\phi_p u_p + V\phi_e u_e + V_{v_p}\dot{v}_p + V_{v_e}\dot{v}_e + \psi(u_p, u_T, u_e, v_p, v_e) \tag{18}$$

The constraints [Eqs. (13) and (15)] can be introduced by either the multiplier[4] or penalty[5] technique; therefore, we let the nonlinear function ψ represent the constraints. The examples below have been run with the penalty method implemented.

In forming the adjoint equations, we observe that the positions do not appear in Eq. (18). Hence the corresponding adjoint variables are constant and equal to the transversality values, which are determined by the final line-of-sight direction from pursuer to evader, θ_f, as given by

$$V_{x_p} = -V_{x_e} = -\cos\theta_f \tag{19}$$

$$V_{y_p} = -V_{y_e} = -\sin\theta_f \tag{20}$$

Then the adjoint equations for the headings and velocities can, in accordance with Ref. 4, be expressed in the form

$$\dot{V}_{\phi_p} = v_p\sin(\theta_f - \phi_p), \qquad V_{\phi_p}(t_f) = 0 \tag{21}$$

$$\dot{V}_{\phi_e} = -v_e\sin(\theta_f - \phi_e), \qquad V_{\phi_e}(t_f) = 0 \tag{22}$$

$$\dot{V}_{v_p} = \cos(\theta_f - \phi_p) - V_{v_p}\left(T_{p_{v_p}}u_T - D_{p_{v_p}}\right)/m_p - \psi_{v_p}, \qquad V_{v_p}(t_f) = 0 \tag{23}$$

$$\dot{V}_{v_e} = -\cos(\theta_f - \phi_e) - V_{v_e}\left(T_{e_{v_e}} - D_{e_{v_e}}\right)/m_e - \psi_{v_e}, \qquad V_{v_e}(t_f) = 0 \tag{24}$$

The optimal controls are found in terms of the Hamiltonian as

$$\left(u_p^*, u_T^*\right) = \arg\min_{u_p, u_T}\{H\} \tag{25}$$

$$u_e^* = \arg\max_{u_e}\{H\} \tag{26}$$

By satisfying Eqs. (1-8), (13), (15), and (21-26), a stationary solution to the pursuit-evasion game is reached. However, those equations are not solved easily. An excellent way to solve such an optimal control problem numerically is to use the DDP method.

Numerical Solution

The optimization problem is solved by a DDP algorithm of first order and the CCP technique. The DDP algorithm is an iterative procedure, and convergence difficulties are almost certain to occur. In order to control the convergence, we define a reformulated Hamiltonian as

$$\tilde{H} = H + \tfrac{1}{2}c_1(u_p - \bar{u}_p)^2 + \tfrac{1}{2}c_2(u_T - \bar{u}_T)^2 - \tfrac{1}{2}c_3(u_e - \bar{u}_e)^2 \tag{27}$$

where c_1, c_2, and c_3 are convergence control parameters and the overbar indicates variables from the previous iteration. $\tilde{H}$ is used in place of H in Eqs. (25) and (26), and by properly chosen c_i values an acceptable convergence course can be established. For further details about the DDP and CCP implementations see Ref. 4.

Interpretation of the Adjoint Variables

Consider first the general case by letting x represent the state vector. The interpretation of the vector V_x can be visualized by expanding the cost functional to second order as

$$V(x + \delta x; t) = V(x; t) + V_x^T\delta x + \tfrac{1}{2}\delta x^T V_{xx}\delta x \tag{28}$$

This means that a change δx in the trajectory at t gives a cost change. Now, let t equal t_0; then, δx_o corresponds to the change in the initial state. Considering a nondifferential game problem, an optimal initial state is such that $V_x(x_0; t_0)$ must vanish and $V_{xx}(x_0; t_0)$ must be positive or negative semidefinite for minimum or maximum problems, respectively. In searching for a stationary value of $V_x(x_0; t_0)$ we may expand $V_x(x_0 + \delta x_0; t_0)$ to first order,

$$V_x(x_0 + \delta x_0; t_0) = V_x(x_0; t_0) + V_{xx}(x_0; t_0)\delta x_0 \tag{29}$$

Then, the best change in δx_0 is

$$\delta x_0 = -V_{xx}^{-1}(x_0; t_0)V_x(x_0; t_0) \tag{30}$$

Using a first-order DDP method, V_{xx}^{-1} is not available and must be replaced by $\epsilon > 0$ for a minimization problem (for maximization, $\epsilon < 0$). In practice several DDP iterations with fixed x_0 have to be run in order to obtain a representative V_x. Then a small step δx_0, according to Eq. (30), can be introduced and some further DDP iterations run with fixed x_0, etc., until $V_x(x_0; t_0)$ is close to zero to a practical level of accuracy.

This procedure is used as a special case in several examples where the best initial velocity v_0 is found for one player. In the case where both players are searching for an optimal x_0, a game saddle-point condition in x_0 must be satisfied.

Results

Several cases have been worked through to examine the importance of an optimal choice of the throttle setting and the pursuer's initial speed, and to study the turning capability in general at low vs high altitudes. The results show that the cost value can be reduced by several kilometers by proper setting of the throttle or the initial speed compared to just using optimal turning. The limitation to the horizontal plane is not serious for the cases with short t_f. A related three-dimensional test with the program in Ref. 7 gave a near-horizontal turn as optimal using $t_f = 20$ s and $v_p(0) = v_{max}$.

The study primarily concerns the movements of the pursuer, and the type of evader is of minor importance as long as his best performance is in the differential game situation. The evader is a moderately good aircraft, and as pursuer an aircraft of today with very high performance is used (see Fig. 2). The selection of the pursuer's initial velocity and the use of the throttle setting have proved to be more crucial for a high-performance aircraft than for an earlier type of aircraft.[8]

The Basic Setup

Some cases demanding a hard turn are shown in Fig. 4, where the two aircraft are assumed to start the game after a near head-on encounter. A 10-deg departure from an exact head-on encounter is assumed in order to create a unique situation. Otherwise there is a choice for both aircraft of doing either a left or right turn. Refer to the differential game discussion in Ref. 7 where several stationary solutions and their saddle-point conditions have been examined. The saddle-point verification is not discussed herein and it is assumed that the correct differential game saddle point is the one shown. In this particular case study a final time of 20 s is used in order to model the situation in which the pursuer wants to turn around and launch a missile. The pursuer's maximum speed and corner speed are 409 and 200 m/s, respectively.

Three relevant cases are studied as shown in Fig. 4. Case 1 is optimal turning with forced full throttle. Case 2 has, in addition, an optimal throttle setting, throttle off for the first 11 s, with the switch time being denoted by t_{sw} below. These cases have $v_0 = v_{max}$. Case 3 is similar to case 1, with $v_0 = 204$ m/s chosen optimally. This choice is an option included in the DDP algorithm [see Eq. (30)]. However, in a practical combat situation it is not always possible to have v_0 at one's disposal as a control parameter. In case 3 the optimal throttle setting coincides with the maximum setting throughout the duration of the game. All of the cases have the common property that the turning is hard until almost in a tail chase position. The worst situation for the pursuer is when the initial speed is maximal and the throttle is at maximum throughout the game. If the initial speed can be chosen optimally, the best situation for the pursuer can be reached. Therefore, let Δr_f denote the difference in cost values for the best and worst cases. In Table 1 it can be seen that the final range r_f can be decreased by 5 km if the initial speed is optimal.

The pursuit-evasion problem is also analyzed at different altitudes. The results for 6.6 and 12 km are shown in Tables 2 and 3. Note that the maximum speed, also used as the initial velocity for the pursuer in some cases, increases with altitude. The evader starts at all altitudes with $v_0 = 250$ m/s.

At an altitude of 6.6 km the detrimental influence of the higher v_0 can be seen clearly. A large difference between worst and best final ranges is achieved. On the other hand, an altitude of 12 km gives a very small difference in Δr_f when using throttle-off or optimally chosen v_0 (see Table 3). This might be attributed to the increased induced drag which makes the speed loss very rapid (see Fig. 5), with the speed dropping from $v_{max} = 604$ m/s to somewhere around 280 m/s. The speed drops to about the same value for all three cases in Table 3. Also note the algorithm's ability to resolve the sharp throttle switch. The time lag in throttle setting is neglected herein. This is a simplification of the real problem. Typical optimal trajectories for this study are shown in Fig. 6.

In order to obtain a consistent comparison between different altitudes a t_f of 27 s was chosen. This t_f was chosen in the sense of having a similar off-boresight angle to the evader at $t = t_f$ for the pursuer when using the maximal speed as the initial velocity. In picking out a few altitudes Fig. 7 was obtained. There is an altitude region where the choice of the initial velocity to enter the game is important. (If the choice is not possible, the throttle should be shut off.) The optimal initial velocity is depicted also, and is seen to rise up to maximum speed when the altitude increases. Above an altitude of 14 km the evader is not able to fly. The weakness of the evader will affect the results even earlier; therefore, the cases above 12-13 km will be of minor significance.

Table 2 Results for several initial velocities (altitude = 6.6 km, t_f = 35 s)

v_0	t_{sw}	r_f	Comments
409	5.6	4.95	$v_0 = v_{max}$ at sea level
573	10.2	7.81	$v_0 = v_{max}$ at 6.6 km
322	0.0	4.38	v_0 optimally chosen
573	0.0	9.99	t_{sw} forced full throttle

Table 3 Results for several initial velocities (altitude = 12 km, t_f = 40 s)

v_0	t_{sw}	r_f	Comments
604	5.6	8.53	Maximum v_0
604	0.0	8.81	t_{sw} forced full throttle
497	0.0	7.98	Optimally chosen v_0

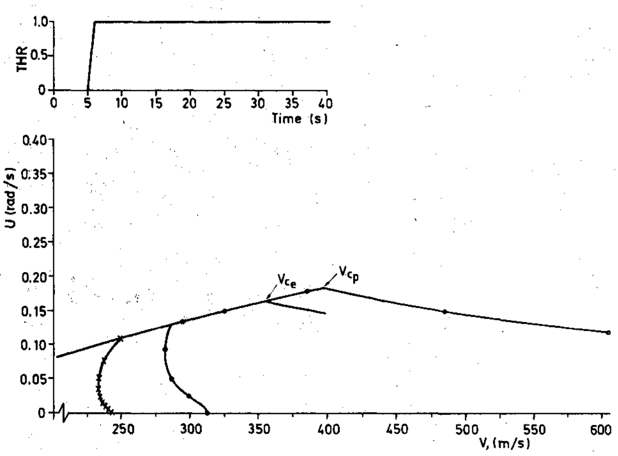

Fig. 5 Optimal controls vs time and velocity, respectively, with throttle off at 12 km with t_f = 40 s (5 s between marks).

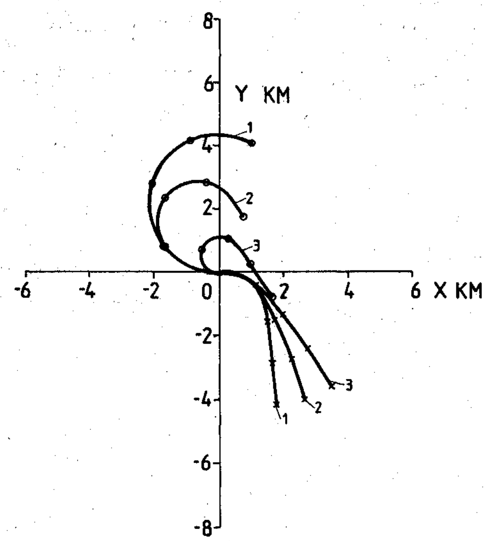

Fig. 4 Optimal trajectories at sea level, cases 1-3 (5 s between marks).

Table 1 Results from runs at sea level, with t_f = 20 s

Case	v_0	t_{sw}	r_f	Δr_f
1	409	0	8.37	
2	409	11	6.11	5.03
3	204	0	3.34	

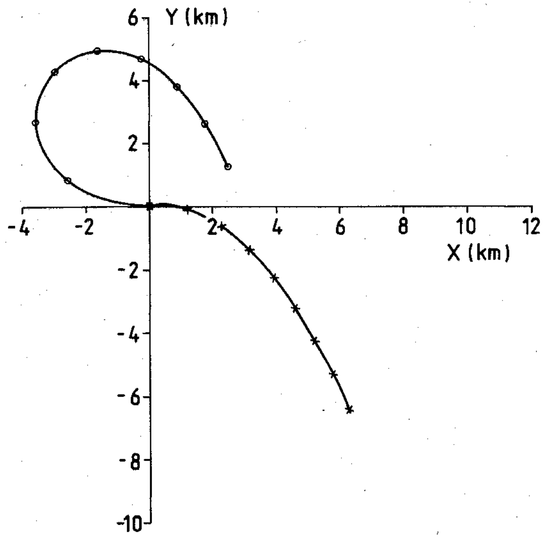

Fig. 6 Optimal trajectories using optimal v_0 at 12 km with $t_f = 40$ (5 s between marks).

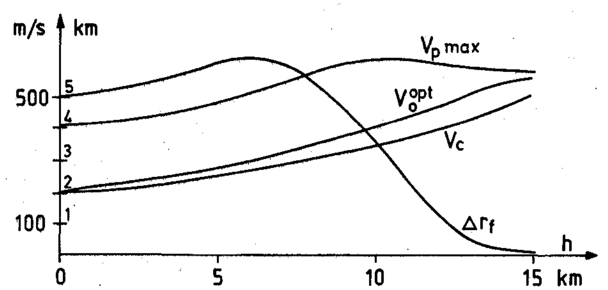

Fig. 7 Maximal velocity, corner speed, optimally chosen initial velocity for pursuer, and span in cost value vs altitude, all at $t_f = 27$ s.

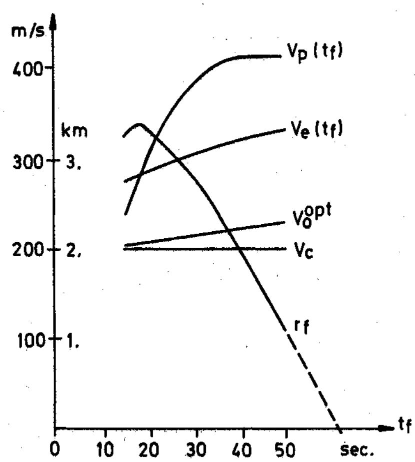

Fig. 8 Final velocities, v_0^{opt} and r_f at sea level as functions of t_f when pursuer's initial velocity is v_0^{opt}.

Effect of Various Values of t_f

The effect of various values of t_f at sea level was investigated also (see Fig. 8). The pursuer is fast and has a very good acceleration performance. Therefore, he will catch up with the evader at about $t_f = 65$ s. However, note that under certain conditions the tail chase part can have a solution in accordance with Ref. 2, where the problem of side-stepping is analyzed as a final evasive maneuver. From Fig. 8 we learn that v_0 shall be inside a certain band, where the lower limit is the corner speed. Also, observe that if t_f is chosen too low there will be the trivial solution $v_0 = 0$. The optimal range reaches a peak value around $t_f = 18$ s, because the pursuer's final speed below this point is less than that of the evader.

Conclusions

A first-order modified DDP method has been shown to be satisfactory in solving a nonlinear differential game problem. Using fixed final time as a game parameter, several combat situations are modeled. The optimal throttle control is bang-bang in nature. However, the algorithm used has solved this problem successfully. For cases demanding a hard turn using a high-performance aircraft, the throttle control is important.

The results point out the importance of having the ability to optimize a nonlinear problem. Concerning the new generation of high-performance aircraft used as the pursuer in this paper we can claim the following. In some situations, having the throttle off during the first few seconds can exert a considerable influence on the optimal cost value. This could mean that the evader will be well positioned within the launch boundary for the pursuer's missile. Furthermore, it is even more important to be able to choose the optimal initial velocity for the different tactical situations that may occur. Good acceleration performance therefore should be combined with adequate deceleration means, such as airbrakes, in order to have the initial velocity at one's disposal. The high performance can cause a bad situation for the aircraft if poorly managed, particularly at altitudes below 8-10 km.

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